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Abstract. We study the approximation in Lebesgue spaces of wavelet bi-frame systems given by translations and dilations of a finite set of generators. A complete characterization of the approximation spaces associated with best m-term approximation of wavelet bi-framelet systems is given. The characterization depends, in general, on the number of vanishing moments of the generators. However, in some cases it is possible to get rid of this restriction, but at the price of replacing the canonical expansion by another linear expansion. This is done by expanding a suitable wavelet in the wavelet bi-frame system.

§1. Introduction

Given a finite collection of functions $\Psi = \{\psi^1, \psi^2, \dots, \psi^L\} \subset L_2(\mathbb{R}^d)$ we use the notation $X(\Psi)$ to denote the corresponding "wavelet" system,

$$X(\Psi) := \{2^{jd/2}\psi^\ell(2^j \cdot -k) \,|\, j \in \mathbb{Z}, k \in \mathbb{Z}^d, \ell = 1, 2, \dots, L\}.$$

A wavelet bi-frame for $L_2(\mathbb{R}^d)$ consists of two sequences of wavelets $\Psi = \{\psi^1, \psi^2, \dots, \psi^L\} \subset L_2(\mathbb{R}^d)$ and $\tilde{\Psi} = \{\tilde{\psi}^1, \tilde{\psi}^2, \dots, \tilde{\psi}^L\} \subset L_2(\mathbb{R}^d)$ for which the systems $X(\Psi)$ and $X(\tilde{\Psi})$ are Bessel systems, and satisfy the perfect reconstruction formula

$$f = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}, k\in\mathbb{Z}^d} \langle f, \tilde{\psi}^\ell_{j,k}\rangle \psi^\ell_{j,k}, \qquad \forall f \in L_2(\mathbb{R}^d), \tag{1}$$

where

$$\psi_{j,k} := 2^{jd/2}\psi(2^j \cdot -k), \qquad j \in \mathbb{Z},\ k \in \mathbb{Z}^d.$$

This definition implies that both $X(\Psi)$ and $X(\tilde{\Psi})$ are frames for $L_2(\mathbb{R}^d)$ and in fact the roles of Ψ and $\tilde{\Psi}$ are interchangeable in (1). The special case with $\Psi = \tilde{\Psi}$ corresponds to a so-called tight wavelet frame.

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The most common method to construct wavelet bi-frames relies on so-called extension principles. The resulting bi-frames are based on a multiresolution analysis, and the generators are often called bi-framelets. The first extension principle is due to Ron and Shen [9, 10] and the more evolved extension principles useful for constructing bi-frames were introduced independently in [3] and [4].

In this paper we discuss two problems related to wavelet bi-frame systems. The first is the nonlinear approximation properties of the systems $X(\Psi)$ and $X(\tilde{\Psi})$ when the approximation error is measured in $L_p(\mathbb{R}^d)$. That is, we consider the (nonlinear) set

$$\Sigma_m(X(\Psi)) := \Big\{ \sum_{i\in\Lambda} c_i g_i \,\Big|\, c_i \in \mathbb{C}, g_i \in X(\Psi), \operatorname{card}\Lambda \leq m \Big\}$$

of all possible m-term expansions with elements from the system $X(\Psi)$. The error of the best m-term approximation to an element $f \in L_p(\mathbb{R}^d)$ is then $\sigma_m(f, X(\Psi))_p := \inf_{f_m\in\Sigma_m(X(\Psi))} \|f - f_m\|_{L_p(\mathbb{R}^d)}$. The problem is to characterize the class $\mathcal{A}_q^\alpha((L_p(\mathbb{R}^d), X(\Psi))$, $\alpha > 0$, $0 < q \leq \infty$, of functions $f \in L_p(\mathbb{R}^d)$ satisfying

$$|f|_{\mathcal{A}_q^\alpha(L_p(\mathbb{R}^d),X(\Psi))} := \bigg(\sum_{m=1}^{\infty} \big(m^\alpha \sigma_m(f, X(\Psi))_p\big)^q \frac{1}{m} \bigg)^{1/q} < \infty,$$

with a suitable modification for $q = \infty$. A (quasi)norm on $\mathcal{A}_q^\alpha$ is given by $\|f\|_{\mathcal{A}_q^\alpha(L_p(\mathbb{R}^d),X(\Psi))} = \|f\|_p + |f|_{\mathcal{A}_q^\alpha(L_p(\mathbb{R}^d),X(\Psi))}$. The problem of characterizing $\mathcal{A}_q^\alpha((L_p(\mathbb{R}^d), X(\Psi))$ was treated by the authors in [2, 1] by proving appropriate Bernstein and Jackson inequalities for wavelet bi-frame systems. We will give an overview of the results in Section 3.

A result of the estimates in Section 3 is that a smooth function (in the Besov sense) has a very sparse canonical expansion in the wavelet bi-frame system provided that the elements in the dual system $\tilde{\Psi}$ have sufficiently many vanishing moments. Unfortunately, most of the common bi-frames only have a few vanishing moments. In order to overcome this obstacle we consider approximation with an oversampled version of the wavelet bi-frame system $X(\Psi)$ in Section 4. For such systems we prove that a Jackson inequality holds independently of the number of vanishing moments of the elements in $\tilde{\Psi}$. This is done by expanding "nice" wavelets in oversampled wavelet bi-frame systems. Moreover, we give an example where such expansion is possible in a non-oversampled bi-frame system.

The structure of the paper is as follows. In Section 2 we review the most common method for constructing wavelet bi-frames using the so-called extension principles. In Section 3 we state the characterization of the approximation space $\mathcal{A}_q^\alpha((L_p(\mathbb{R}^d), X(\Psi))$, $\alpha > 0$, $0 < q \leq \infty$, given in [2]. Finally in Section 4 we study characterizations of the approximation spaces for oversampled wavelet bi-frame systems.

§2. Bi-framelet Systems

In this section we will briefly describe how to construct MRA-based wavelet bi-frames – called bi-framelets – through so-called extension principles. The extension principles will be used in Section 3 when we consider the problem of expanding wavelets in the bi-framelet system. The extension principles to construct bi-frames were introduced independently in [3] and [4]. Below we use the notation of [4].

Let $\boldsymbol{\tau} = (\tau_0, \tau_1, \ldots, \tau_L)$ and $\tilde{\boldsymbol{\tau}} = (\tilde{\tau}_0, \tilde{\tau}_1, \ldots, \tilde{\tau}_L)$ be two sequences of $2\pi\mathbb{Z}^d$-periodic essentially bounded functions. Assume that τ_0 and $\tilde{\tau}_0$ both generate refinable functions

$$\hat{\phi}(2\xi) = \tau_0(\xi)\hat{\phi}(\xi) \qquad \text{and} \qquad \hat{\tilde{\phi}}(2\xi) = \tilde{\tau}_0(\xi)\hat{\tilde{\phi}}(\xi),$$

satisfying

$$\{\phi(\cdot - k)\}_{k\in\mathbb{Z}} \quad \text{and} \quad \{\tilde{\phi}(\cdot - k)\}_{k\in\mathbb{Z}}$$

are Riesz sequences in $L_2(\mathbb{R}^d)$. Here $\hat{\phi}(\xi)$ is the Fourier transform of the function $\phi(x)$. We associate the wavelets to $\boldsymbol{\tau}$ and $\tilde{\boldsymbol{\tau}}$ as follows

$$\hat{\psi}^\ell(2\xi) = \tau_\ell(\xi)\hat{\phi}(\xi), \qquad \hat{\tilde{\psi}}^\ell(2\xi) = \tilde{\tau}_\ell(\xi)\hat{\tilde{\phi}}(\xi).$$

Assuming that the systems $X(\Psi)$ and $X(\tilde{\Psi})$ are both Bessel, we define the mixed fundamental function of the parent vectors $\boldsymbol{\tau}$ and $\tilde{\boldsymbol{\tau}}$ by

$$\Theta(\xi) := \sum_{j=0}^{\infty}\sum_{\ell=1}^{L} \tau_\ell(2^j\xi)\overline{\tilde{\tau}_\ell(2^j\xi)} \prod_{m=0}^{j-1} \tau_0(2^m\xi)\overline{\tilde{\tau}_0(2^m\xi)}.$$

The following theorem proved in [4] is the main tool to create bi-framelet systems, the theorem is called the Mixed Oblique Extension Principle.

Theorem 1. [Mixed OEP] *Let $\boldsymbol{\tau}$ and $\tilde{\boldsymbol{\tau}}$ be the combined mask of the systems $X(\Psi)$ and $X(\tilde{\Psi})$, respectively. Assume that the systems $X(\Psi)$ and $X(\tilde{\Psi})$ are Bessel systems. Suppose there exists a 2π-periodic function Θ satisfying*

a) Θ is essentially bounded, continuous at the origin, and $\Theta(0) = 1$.
b) For every $\xi \in [-\pi, \pi]^d$ and $\nu \in \{0, \pi\}^d$,

$$\Theta(2\xi)\tau_0(\xi)\overline{\tilde{\tau}(\xi+\nu)} + \sum_{\ell=1}^{L} \tau_\ell(\xi)\overline{\tilde{\tau}_\ell(\xi+\nu)} = \begin{cases} \Theta(\xi), & \text{if } \nu = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Then $X(\Psi)$, $X(\tilde{\Psi})$ is a bi-framelet system.

In fact, [4] uses weaker assumptions on the refinable functions than considered here, but we will not use this generalization here.

When $X(\Psi) = X(\tilde{\Psi})$, Theorem 1 gives the so-called Oblique Extension principle, see [4]. If, in addition, $\Theta \equiv 1$, Theorem 1 reduces to the Unitary Extension Principle, see [9, 10].

The reader can consult [3] and [4] for many explicit examples on how to construct framelet systems using the different extension principles.

§3. Nonlinear Approximation with Bi-framelets

Now we consider the problem of characterizing $\mathcal{A}_q^\alpha(L_p(\mathbb{R}^d), X(\Psi))$. To obtain this characterization one needs to prove a Bernstein and Jackson estimate for the bi-framelet system. This problem was investigated in detail in [2]. Here we will only give a summary of the results, and refer to [2] for the details.

Before we state the main result, Theorem 2, we need to introduce the following notation. Let Λ be the function defined by

$$\Lambda(x,p,\gamma/d) := \begin{cases} p(1-x) & \text{for } x \le 1-1/p, \\ (x+1/p)^{-1} & \text{for } 1-1/p < x \le \gamma/d - 1/p, \\ d/\gamma & \text{for } \gamma/d - 1/p < x. \end{cases} \tag{2}$$

Recall that for a function ϕ, the set $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}^d}$ is a locally linearly independent set if the set $\{\phi(\cdot - k)\}_{k\in\Gamma}$ is linearly independent, where

$$\Gamma = \{k \in \mathbb{Z}^d \colon |\{x \in (0,1)^d \colon \phi(x-k) \neq 0\}| > 0\}.$$

We also need the following function class.

Definition 1. *For $N \in \mathbb{N}$ and $\gamma > 0$ we let $M_\gamma^N(\mathbb{R}^d)$ be the set of all functions f defined on $\mathbb{R}^d$ with N vanishing moments and decay,* i.e., *for which*

$$\int_{\mathbb{R}^d} x^\alpha f(x)dx = 0 \qquad \text{for } \alpha \in \mathbb{N}^d,\ |\alpha| < N,$$

and

$$|f(x)| \le C(1+|x|)^{-d-N-\gamma} \qquad \text{for } x \in \mathbb{R}^d. \tag{3}$$

We can now state the theorem.

Theorem 2. *Let $X(\Psi)$, $X(\tilde{\Psi})$ be a bi-framelet system and assume that $X(\Psi)$ is based on a compactly supported refinable function ϕ where:*

1. *$\phi \in W^s(L_\infty(\mathbb{R}^d))$ with $s \ge 0$;*
2. *If $d > 1$, $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}^d}$ is a locally linearly independent set;*
3. *The functions $\tau_\ell(\xi)$ are trigonometric polynomials (see Section 2);*
4. *$\tilde{\Psi} \subset C^\beta(\mathbb{R}^d) \cap M_\gamma^{N_1}(\mathbb{R}^d)$ for some $\beta > 0$, $N_1 \in \mathbb{N}$ and $\gamma > d$.*

Let $p \in (1,\infty)$ and $\tau := (\alpha + 1/p)^{-1}$ where we assume

$$0 < \alpha < \min\left\{\frac{s}{d}, \frac{1}{\Lambda\left(\frac{N_1}{d}\right)} - \frac{1}{p}\right\}, \tag{4}$$

with $\Lambda(x) = \Lambda(x,p,d/\gamma)$ given by (2). Then we have the Bernstein estimate

$$|S|_{B_\tau^{d\alpha}(L_\tau(\mathbb{R}^d))} \leq Cm^\alpha \|S\|_{L_p(\mathbb{R}^d)}, \qquad \forall S \in \Sigma_m(X(\Psi)), \qquad \forall m \geq 1,$$

and the Jackson estimate

$$\sigma_m(f, X(\Psi))_p \leq Cm^{-\alpha}\|f\|_{B_\tau^{d\alpha}(L_\tau(\mathbb{R}^d))}.$$

Moreover, for each $0 < \beta < \alpha$, $q \in (0,\infty]$, we have the characterization

$$\mathcal{A}_q^\beta(L_p(\mathbb{R}^d), X(\Psi)) = \left(L_p(\mathbb{R}^d), B_\tau^{d\alpha}(L_\tau(\mathbb{R}^d))\right)_{\beta/\alpha,q}. \tag{5}$$

To prove Theorem 2, two steps are needed. First the Bernstein estimate is established by using the Bernstein inequality for refinable functions [5, 7] using the fact that each generator from Ψ is a finite linear combination of translates of ϕ. To prove the Jackson estimate, the bi-framelet system is analyzed in a sufficiently nice orthonormal wavelet system yielding information on the "change of basis" matrix. The matrix estimate can then be used to translate the known Jackson estimate for wavelet system to the bi-framelet case. We refer to [2] for the technical details.

§4. Framelet Expansions of Wavelets

In the proof of Theorem 2 in [2] it is essential to require that the dual system $X(\tilde{\Psi})$ has a sufficient number of vanishing moments. The remainder of this paper is devoted to the study of ways of avoiding this requirement since many bi-framelet systems only have few vanishing moments. For example, all systems build using the unitary extension principle of Ron and Shen have only one vanishing moment. The key to avoiding the vanishing moment "trap" is to oversample the system. For technical reasons we will only consider the univariate case in this section. Given a finite collection of functions $\Psi = \{\psi^\ell\}_{\ell=1,2,\ldots,L}$ in $L_2(\mathbb{R})$ and $R \geq 1$ we let $X_R(\Psi)$ denote the oversampled system,

$$X_R(\Psi) := \left\{2^{j/2}\psi^\ell(2^j \cdot -k/R) \colon j,k \in \mathbb{Z}, \ell = 1,2,\ldots,L\right\}.$$

It turns out that if $X(\Psi)$ is a frame in $L_2(\mathbb{R})$ so is the oversampled system $X_R(\Psi)$. Moreover, if $X(\Psi), X(\tilde{\Psi})$ is a "nice" bi-framelet system, the oversampled system $X_{2^N}(\Psi)$, $N \in \mathbb{N}$, again, gives rise to an approximation space no larger than a Besov space. More precisely, suppose condition 1 and 3 in Theorem 2 are satisfied. Then the Bernstein estimate

$$|S|_{B^{d\alpha}_{\tau}(L_\tau(\mathbb{R}^d))} \leq Cm^\alpha \|S\|_{L_p(\mathbb{R})}, \qquad \forall S \in \Sigma_m(X_{2^N}(\Psi)), \quad \forall m \geq 1, \quad (6)$$

holds true for each $0 < \alpha < s$, $0 < p \leq \infty$, and $1/\tau := \alpha + 1/p$, where $C = C(\alpha, p)$.

In order to get a complete characterization of the approximation spaces based on $X_R(\Psi)$ in terms of Besov spaces, we need to prove a matching Jackson estimate. The following result was given in [2].

Proposition 1. *Let $X(\Psi)$, $X(\tilde{\Psi})$ be a wavelet bi-frame system, $X(\eta)$ a bi-orthogonal wavelet basis and $r > 0$ so that the Besov space $B^r_\tau(L_\tau(\mathbb{R}))$, $0 < \tau < \infty$, can be characterized by*

$$\Big\{ f \in L_\tau(\mathbb{R})\colon \sum_{j,k\in\mathbb{Z}} (2^{j(r+1/2-1/\tau)} |\langle f, \eta_{j,k}\rangle|)^\tau < \infty \Big\}. \quad (7)$$

Assume there exists a sequence $\{d^\ell_k\}_{k\in\mathbb{Z},\ell=1,2,\ldots,L} \in \ell_{1/(r+1)}$, such that

$$\eta(x) = \sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}} d^\ell_k \psi^\ell(x - k/R).$$

Then, for $1 < p < \infty$, and $0 < \alpha = 1/\tau - 1/p < r$, we have the Jackson inequality

$$\sigma_m(f, X_R(\Psi))_p \leq Cm^{-\alpha} \|f\|_{B^\alpha_\tau(L_\tau(\mathbb{R}))}. \quad (8)$$

Now, by combining the Bernstein and Jackson inequalities (6) and (8), we obtain a complete characterization of the approximation spaces $\mathcal{A}^\beta_q(L_p(\mathbb{R}^d), X_{2^N}(\Psi))$, $N \in \mathbb{N}$, not limited by the number of vanishing moments of the dual system $\tilde{\Psi}$.

Corollary 1. *Suppose $X(\Psi)$, $X(\tilde{\Psi})$ is a bi-framelet system satisfying all the assumptions of Proposition 1 for $R = 2^N$, together with condition 1 and 3 in Theorem 2. Then, for $0 < \alpha < \min\{s, r\}$, $0 < \beta < \alpha$, $q \in (0, \infty]$, we have the characterization*

$$\mathcal{A}^\beta_q(L_p(\mathbb{R}^d), X_{2^N}(\Psi)) = \big(L_p(\mathbb{R}^d), B^\alpha_\tau(L_\tau(\mathbb{R}))\big)_{\beta/\alpha,q}. \quad (9)$$

In our strategy to get a Jackson inequality for the (oversampled) framelet system $X_R(\Psi)$, the crucial issue is to identify some "nice" wavelet(s) that can be expanded sparsely in terms of the oversampled bi-frame system. For spline-based *tight* framelets, it was shown in [6] how to get a finite expansion of a nice semi-orthogonal wavelet in the twice oversampled ($R = 2$) framelet system. Let us consider a construction valid for more general bi-framelets. We let $\mathrm{VM}(\Psi)$ denote the number of vanishing moments of the functions in Ψ, i.e., each function in Ψ has $\mathrm{VM}(\Psi)$ vanishing moments, and at least one function in Ψ fails to have $\mathrm{VM}(\Psi)+1$ vanishing moments.

In Proposition 2 below we will refer to the following standard wavelet associated to a scaling function.

Definition 2. *Let ϕ be a univariate scaling function generated by the refinement filter $\tau_0(\xi)$. Let $P(\xi) := \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi-k)|^2$, and define the filter*

$$\tau^s(\xi) = \frac{e^{-i\xi}\overline{\tau_0(\xi+\pi)}\sqrt{P(\xi+\pi)}}{\sqrt{P(2\xi)P(\xi)}}. \tag{10}$$

The "standard" orthonormal wavelet η associated with the scaling function ϕ is defined by $\hat{\eta}(2\xi) := \tau^s(\xi)\hat{\phi}(\xi)$

Proposition 2. *Let $X(\Psi)$, $X(\tilde{\Psi})$ be an MRA-based wavelet bi-frame system and let ϕ and η be respectively the scaling function and the associated standard orthonormal wavelet. Suppose that each filter τ_ℓ, $\ell = 0,1,\ldots,L$, is a trigonometric polynomial,*

$$\sum_{\ell=1}^{L} |\tau_\ell(\xi)|^2 > 0, \quad \text{for } \xi \neq 0,$$

$\mathrm{VM}(\Psi) \le \mathrm{VM}(\eta)$, and ϕ is an r-regular scaling function (not necessarily orthonormal). Then there exists $\{d_k^\ell\} \in \bigcap_{\tau>0} \ell_\tau$ such that

$$\eta(x) := \sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}} d_k^\ell \psi^\ell(x-k/2).$$

Proof: We want to expand the standard orthonormal wavelet η given by Definition 2 in the twice oversampled framelet system. In the frequency domain the problem is to find "nice" 2π-periodic functions $Q_\ell(\xi)$ such that

$$\hat{\eta}(\xi) = \sum_{\ell=1}^{L} Q_\ell(\xi/2)\tau_\ell(\xi/2)\hat{\phi}(\xi/2).$$

We will look for Q_ℓ of the form $Q_\ell(\xi) = Q(\xi)\overline{\tau_\ell(\xi)}$. Using Definition 2, we see that the problem will be solved if Q_ℓ has fast decaying Fourier coefficients and Q satisfies

$$Q(\xi)\sum_{\ell=1}^{L} |\tau_\ell(\xi)|^2 = \frac{e^{-i\xi}\overline{\tau_0(\xi+\pi)}}{\sqrt{P(\xi)}}.$$

Hence, we define for $\xi \neq 0$

$$Q_\ell(\xi) := \frac{\tau_\ell(\xi) \cdot \overline{\tau_0(\xi+\pi)}}{\sum_{\ell=1}^{L} |\tau_\ell(\xi)|^2} \cdot \frac{e^{-i\xi}}{\sqrt{P(\xi)}}. \tag{11}$$

Let us check that Q_ℓ can be readily extended at $\xi = 0$ and that the resulting extension has no pole on the unit circle. First, we have, for ξ close to zero, $\sum_{\ell=1}^{L} |\tau_\ell(\xi)|^2 \asymp |\xi|^{2\cdot\mathrm{VM}(\Psi)}$. Then, we use the fact that, for ξ close to zero,

$$|\tau_\ell(\xi)\overline{\tau_0(\xi+\pi)}| = O(|\xi|^{\mathrm{VM}(\Psi)+\mathrm{VM}(\psi)}) = O(|\xi|^{2\cdot\mathrm{VM}(\Psi)}).$$

We conclude by proving that the Fourier coefficients of Q_ℓ decay faster than any polynomial. Notice that $P(\xi)^{-1/2}$ is C^∞ (see, *e.g.*, [8]) so its Fourier coefficients decay faster than any polynomial. The factor

$$\frac{\tau_\ell(\xi) \cdot \overline{\tau_0(\xi+\pi)}}{\sum_{\ell=1}^{L} |\tau_\ell(\xi)|^2}$$

in (11) is a quotient of two trigonometric polynomials with no pole on the unit circle, so its Fourier coefficients decay exponentially which can be seen from its Laurent expansion. □

4.1. Wavelets build with no oversampling

In this subsection we extend the result from Proposition 2 by showing that in certain cases no oversampling is needed to build a wavelet out of the framelet system. The following lemma is a restatement of the oblique extension principle, see [4].

Lemma 1. *Let $X(\Psi)$ be a tight framelet system with associated trigonometric polynomials τ_ℓ, $\ell = 0, 1, \ldots, L$. Define $m_\ell(e^{-i\xi}) := \tau_\ell(\xi)$. Then there exists a rational function $T(z)$, with $T(1) = 1$, such that*

$$m_0(z)\overline{m_0(z^{-1})}T(z^2) + \sum_{\ell=1}^{L} m_\ell(z)\overline{m_\ell(z^{-1})} = T(z), \tag{12}$$

and

$$m_0(z)\overline{m_0(-z^{-1})}T(z^2) + \sum_{\ell=1}^{L} m_\ell(z)\overline{m_\ell(-z^{-1})} = 0. \tag{13}$$

Let $G(z) = \gcd\{m_\ell(z) \colon \ell = 1, 2, \ldots, L\}$. From Lemma 1 we can deduce the following result.

Theorem 3. *Suppose* $m_0(z) = \left(\frac{1+z}{2}\right)^r R(z)$, *for some* $r \in \mathbb{N}$, *where* $|R(1)| = 1$ *and* $R(-1) \neq 0$. *Then* $G(z) = (1-z)^n \tilde{G}(z)$ *for some* $1 \leq n \leq r$, *where* $\tilde{G}(1) \neq 0$.

Proof: By (12) we have

$$\sum_{\ell=1}^{L} |m_\ell(1)|^2 = 1 - |m_0(1)|^2 = 1 - |R(1)|^2 = 0.$$

Thus $(1-z)|G(z)$, i.e., $n \geq 1$.

Now, suppose the lemma holds for some $n \geq 1$. Then (13) yields $(1-z)^n(1+z^{-1})^n = (-1)^n(1+z)^n(1-z^{-1})^n$ is a factor of $m_0(z)m_0(-z^{-1})\cdot T(z^2)$. Since $T(1) = 1$, $R(1) \neq 0$ and $R(-1) \neq 0$, this implies $n \leq r$. $\square$

Let ϕ be a scaling function generated by the refinement filter $\tau_0(\xi)$. Define the "standard" wavelet filter $\tau^s(\xi)$ by (10), and let $m^s(e^{-i\xi}) = \tau^s(\xi)$. Notice that if $m_0(z) = \left(\frac{1+z}{2}\right)^r R(z)$ for some $r \in \mathbb{N}$, then $(1-z)^r$ is a factor of $m^s(z)$.

Recall that any bi-orthogonal wavelet η associated with the scaling function ϕ is given by

$$\hat{\eta}(2\xi) = v(2\xi)\tau^s(\xi)\hat{\phi}(\xi) \tag{14}$$

for some 2π-periodic function v with $|v| = 1$ a.e..

Let us try to expand such a wavelet in terms of integer shifts of the framelets ψ^ℓ. A *necessary and sufficient condition* for such an expansion to be possible is the existence of polynomials $p^\ell(z)$ such that

$$\hat{\eta}(\xi) = \sum_{\ell=1}^{L} p^\ell(e^{-i\xi})\hat{\psi}^\ell(\xi).$$

Using (14) this is equivalent to the existence of polynomials $p^\ell(z)$ such that

$$\sum_{\ell=1}^{L} p^\ell(e^{-i2\xi})\tau_\ell(\xi) = v(2\xi)\tau^s(\xi). \tag{15}$$

According to Theorem 3 we have $m_\ell(z) = (1-z)^n \cdot (a^\ell(z^2) + zb^\ell(z^2))$ for some $1 \leq n \leq r$. Likewise we can write $m^s(z) = (1-z)^n(A(z^2) + zB(z^2))$. Thus, with $q(e^{-i\xi}) := v(\xi)$, (15) can be rewritten

$$q(z)\begin{bmatrix} A(z) \\ B(z) \end{bmatrix} = \begin{bmatrix} a^1(z) & \cdots & a^L(z) \\ b^1(z) & \cdots & b^L(z) \end{bmatrix} \begin{bmatrix} p^1(z) \\ \vdots \\ p^L(z) \end{bmatrix}. \tag{16}$$

Now, using Proposition 1 we obtain the following result.

Theorem 4. *Let $X(\Psi)$ be a tight framelet system based on a compactly supported refinable function $\phi \in W^s(L_\infty(\mathbb{R}))$, $s \geq 0$, and trigonometric polynomials $\tau_\ell(\xi)$, $\ell = 0, 1, \ldots, L$. Suppose the wavelet η given by (14) can be used to characterize the Besov space $B^r_\tau(L_\tau(\mathbb{R}))$, $r > 0$, as given in (7), and suppose there exist polynomials $\{p^\ell(z)\}_{\ell=1,2,\ldots,L}$ such that (16) is satisfied. Let $p \in (1, \infty)$, $\alpha < \min\{s, r\}$, and $\tau := (\alpha + 1/p)^{-1}$. Then, for each $0 < \beta < \alpha$ and $q \in (0, \infty]$, we have the characterization*

$$\mathcal{A}^\beta_q(L_p(\mathbb{R}), X(\Psi)) = (L_p(\mathbb{R}), B^\alpha_\tau(L_\tau(\mathbb{R})))_{\beta/\alpha,q}\,.$$

4.2. The case $L = 2$

Let us consider the case $L = 2$. Any solution of the 2×2 case will also satisfy

$$q(z)\begin{bmatrix} C(z) \\ D(z) \end{bmatrix} := q(z)\begin{bmatrix} b^2(z) & -a^2(z) \\ -b^1(z) & a^1(z) \end{bmatrix}\begin{bmatrix} A(z) \\ B(z) \end{bmatrix} = \Delta(z)\begin{bmatrix} p^1(z) \\ p^2(z) \end{bmatrix} \tag{17}$$

with $\Delta(z) := a^1(z)b^2(z) - b^1(z)a^2(z)$. Let us write $\Delta(z) = \tilde{\Delta}(z)\Delta_0(z)$ where $\tilde{\Delta}(z)$ has no zero on the unit circle while all zeros of Δ_0 are on it (or Δ_0 might be identically 1).

The following proposition was considered in [6] in the less general case of spline based MRAs. We give the proof for the sake of completeness.

Proposition 3. *There is a solution $p^1(z), p^2(z), q(z)$ to (16) with $q(z) \neq 0$, $|z| = 1$ if and only if $\Delta_0(z)$ divides $\gcd\{C(z), D(z)\}$.*

Proof: Let $p^1(z), p^2(z), q(z)$ be such a solution. By hypothesis $q(z)$ has no zeroes on the unit circle while $\Delta_0(z)$ has all (if any) of its zeroes on the unit circle, so $\gcd\{q, \Delta_0\} = 1$. However, $\Delta_0 \big| \gcd\{q \cdot C, q \cdot D\}$ and it follows easily that $\Delta_0 \big| \gcd\{C, D\}$. Assuming $\Delta_0(z)$ divides $\gcd\{C(z), D(z)\}$, we can take

$$\begin{aligned} p^1(z) &:= C(z)/\Delta_0(z) \\ p^2(z) &:= D(z)/\Delta_0(z) \\ q(z) &:= \tilde{\Delta}(z). \ \square \end{aligned}$$

Let us now consider two examples of framelet systems. The first example shows that (16) has a solution in some cases. The second example will show that there are framelet systems for which (16) has no solution.

Example 1. [4, Example 2.16] *Let $m_0(z) := (1-z)^2/4$. It can be verified that the masks $m_1(z) := -(1-z)^2/4$, and $m_2(z) := -\sqrt{2}(1+z)(1-z)/4$ generate a tight framelet system associated with $m_0(z)$. One can easily check that*

$$\begin{bmatrix} a^1(z) & a^2(z) \\ b^1(z) & b^2(z) \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} 1 & \sqrt{2} \\ -1 & \sqrt{2} \end{bmatrix},$$

so $\Delta(z) = -\sqrt{2}/2$. Let η be any bi-orthogonal wavelet associated with m_0, then Proposition 3 shows that (16) has a solution in this case, i.e., that η can be written as a linear combination of integer *shifts of the framelets. In particular, let η be the Chui-Wang wavelet with corresponding mask $m(z) = (1-z)^2(1-4z+z^2)/24$. Since in this case $A(z) = (1+5z)/24$, and $B(z) = -(5+z)/24$, we obtain after a few calculations $p_1(z) = -\frac{1}{2}(1+z)$, and $p_2(z) = \frac{\sqrt{2}}{6}(1-z)$, i.e.,*

$$\eta(x) = -\frac{1}{2}\psi^1(x) - \frac{1}{2}\psi^1(x-1) + \frac{\sqrt{2}}{6}\psi^2(x) - \frac{\sqrt{2}}{6}\psi^2(x-1).$$

Example 2. [4, Example 2.18] *Again, let $m_0(z) := (1-z)^2/4$, and let η be the Chui-Wang wavelet with corresponding mask $m(z) = (1-z)^2(1-4z+z^2)/24$. The masks $m_1(z) := -(1-z)^2/4$, and $m_2(z) := -\sqrt{\frac{6}{24}}(1-z)^2(z+4z^2+z^3)$ also generate a tight framelet. It is easy to obtain*

$$\begin{bmatrix} a^1(z) & a^2(z) \\ b^1(z) & b^2(z) \end{bmatrix} = \begin{bmatrix} 1 & 4z \\ 0 & 1+z \end{bmatrix},$$

$A(z) = (1+z)/24$, and $B(z) = -1/6$. Consequently, $\Delta(z) = 1+z$ with $\tilde{\Delta}(z) = 1$ and $\Delta_0(z) = 1+z$. Also, $C(z) = (1+z)^2/24 + 2/3z$ and $D(z) = -1/6$ so $\gcd\{C, D\} = 1$ which implies that Δ_0 does not divide $\gcd\{C, D\}$ and the expansion of the type (16) is not *possible for this framelet system.*

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